Indian Statistical Institute
B. Math. Hons. II Year
Semestral Examination 2002-2003
Analysis IV

Date:24-04-2003 Total Marks: 50 Instructor: S. Thangavelu

1. Find the general solution of the equation

$$(1-e^x)y''(x)+\frac{1}{2}y'(x)+e^x y(x)=0$$

near the singular point $x=0$. [4]

2. Find the radius of convergence of the power series solution $y(x)=\sum_{n=0}^{\infty} a_n x^n$ of the equation [4]

$$x^2y''(x)+(3x-1)\ y'(x)+y(x)=0.$$

3. Find the power series solution of the equation $y'(x)=1+y(x)^2$ which satisfies the condition $y(0)=0$. [4]

4. Find the general solution of the equation [4]

$$y''(x)-f(x)\ y'(x)+(f(x)-1)\ y(x)=0.$$

5. Prove or disprove the following statement: If y_1 and y_2 are two solutions of the equation $y''(x)+p(x)\ y'(x)+q(x)\ y(x)=0$ on $[a,b]$ where $p,q\in C[a,b]$ and have a common zero in $[a,b]$ then $y_1=c\ y_2$ for some c. [6]

6. If $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ are the Legendre polynomials (where $n=0,1,2,\ldots$) then show that

$$\frac{d}{dx}((1-x^2)P'_n(x))+n(n+1)\ P_n(x)=0.$$

Hence or otherwise show that [4]

$$\int_{-1}^{1} P_n(x)\ P_m(x)\ dx=0 \text{ for } m\neq n.$$

7. Assuming the generating function identity

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} P_k(x)t^k$$

valid for $|x| < 1, |t| < 1$ show that [6]

$$\int_{-1}^{1} P_n(x)^2 dx = 2(2n+1)^{-1}.$$

8. If $p(x)$ is a polynomial of degree $n \geq 1$ such that

$$\int_{-1}^{1} x^k p(x)\ dx = 0, k = 0, 1, 2, \ldots, n-1$$

show that $p(x)$ is a constant multiple of $P_n(x)$. [4]

9. Show that

$$\varphi_\lambda(z) = \int_0^{2\pi} e^{i\lambda\ Re(ze^{i\theta})} d\theta, \quad z \in \mathbb{C}$$

is radial and the function $\varphi_\lambda(r)$ satisfies an ordinary differential equation. Hence or otherwise prove: $\varphi_\lambda(z) = c\ J_0(\lambda|z|)$ for some constant c. [6]

10. Prove the formulas

$$\frac{d}{dx} J_0(x) = -J_1(x), \frac{d}{dx}(xJ_1(x)) = xJ_0(x)$$

and use them to show that the positive zeros of J_0 and J_1 alternate.

[4]

11. Assume that for well behaved functions g on [0,1]

$$\frac{1}{2}\int_0^1 x^{p+1} g(x)\ dx = \sum_{n=1}^{\infty} \frac{1}{\lambda_n J_{p+1}(\lambda_n)} \int_0^1 x\ g(x)\ J_p(\lambda_n x) dx$$

where λ_n are the positive zeros of $J_p(x)$. Using the formula

$$\frac{d}{dx}(x^p J_p(x)) = x^p J_{p-1}(x)$$

show that

$$\sum_{n=1}^{\infty} \lambda_n^{-2} = \frac{1}{4}(p+1)^{-1}, \sum_{n=1}^{\infty} \lambda_n^{-4} = \frac{1}{16}(p+1)^{-2}(p+2)^{-1}.$$

Evaluate: $\sum\limits_{n=1}^{\infty} n^{-2}$ and $\sum\limits_{n=1}^{\infty} n^{-4}$. [6]